Tchuka Ruma Solitaire

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The Game of Tchuka Ruma

The solitaire game "Tchuka Ruma" is described as follows in the book of mathematical diversions by Degrazia [1948, 102–103]:

> **185.** Tchuka Ruma
>
> Tchuka Ruma is an East Indian game which can be played with the most primitive equipment, such as a number of holes and some pebbles. It may be mathematically analyzed, although its theory has not yet been fully explored.
>
> You have a board with five holes in a row. There are two pebbles (or pieces) in each of the first four holes, while the last, called Ruma, is empty:
>
> | I | II | III | IV | R |
> |---|---|---|---|---|
> | 2 | 2 | 2 | 2 | 0 |
>
> The object of the game is to put all the pieces in the Ruma hole in the manner prescribed. The game is begun by taking two pieces from any one hole and placing them, singly, into the next two holes to the right. If you still hold a piece—or, in later moves, more than one—in your hand after you have dropped one in the last hole, the Ruma, you put the next piece in the first hole, at the far left, and from there proceed in your distribution to the right, as usual. If the last piece is dropped in the Ruma you may select any hole for the next distribution. Otherwise you are supposed to empty the hole into which you dropped the last piece, provided it already contains one or more pieces. You may not empty the Ruma. If the last piece out of a hole happens to go into an empty hole, the game is lost. The game is won, on the other hand, when all 8 pieces are in the Ruma.

Degrazia gives as solution [1948, 158]:

2	2	2	2	0		0	1	1	1	5
2	2	0	3	1		0	1	1	0	6
3	3	0	0	2		0	0	2	0	6
3	0	1	1	3		0	0	0	1	7
3	0	0	2	3		0	0	0	0	8
4	0	0	0	4						

185—The distribution of the pieces has to take place in this order: This sequence of moves represents the only possible solution of the four-hole Tchuka Ruma. There is no possible solution with 6 holes and a Ruma hole, and with 3 pieces in each hole. But the more complicated variant of 8 holes and a Ruma hole, and 4 pieces in each hole, has a solution.

Why Is It Interesting?

From a mathematical point of view, Tchuka Ruma is interesting because it is not easy to determine whether it is possible for the player to win a game with a particular number of holes and starting position. Nevertheless, a mathematical analysis should be able to tell us many, if not all, of the winnable and nonwinnable games.

From an anthropological point of view, Tchuka Ruma is interesting because, although there are hundreds of versions of two-player games of the mancala type [Russ 1984], Tchuka Ruma is one of very few solitaire versions, most of which are contemporary inventions [Russ 1992; Gardner 1994].

The other solitaire versions are

- Ise-Ozin-Egbe, from Nigeria, described by Bell and Cornelius [1988, 32, 37, 110];
- an unnamed game, with no provenience given, posed as a problem by Cipra [1992], with solutions by Litchfield [1993] and Callan [1993];
- an unnamed game, with no provenience given, described by Bell and Cornelius [1988, 38, 111]; and
- Tchoukaillon, Monokalah, and Multitchouka, all variations on Tchuka Ruma invented by Véronique Gautheron [Deledicq and Popova 1977, 180–187].

The first two are not games of strategy but mechanical pastimes, in which the player deterministically transfers seeds until the starting array recurs. The third game has the same goal, but the player may sow from any hole and the objective is to achieve the goal in as few sowings as possible.

We have used Tchuka Ruma in Beloit College's liberal-arts mathematics course in Ethnomathematics, as an example of a cultural game that involves mathematical thinking. It is one of the few authentic games that lends itself to complete analysis (for small initial configurations) at this level. We have also used it in Beloit's first course in computer science, asking students to write a program to implement it as a computer game; the game lends itself well to illustrating several fundamental ideas in programming. Using the game with our students, as well as programming it ourselves, led us to discover curious aspects of the game, with some remaining open questions as well.

When Can You Win?

The immediate mathematical question is:

Let n be the number of holes apart from the Ruma and let k the number of seeds in each hole in the starting position. For what numbers n and k can the player win?

Before reading further, the reader should experiment by playing the game for a few different combinations of small numbers of holes and small numbers of seeds.

We use certain terminology in discussing and analyzing the game:

- A *sowing* consists of picking up the seeds in a non-Ruma hole and placing one in each successive hole, until the seeds are exhausted.
- If the player reaches the Ruma with extra seeds in hand, sowing continues into the hole farthest from the Ruma, as if the holes were arranged in a circle. We say that the sowing *wraps* around the Ruma.
- When the last seed in a sowing falls into a nonempty hole that is not the Ruma, the player picks up all the seeds in that hole and sows again. We call this iteration *chaining* of the sowings.
- A *move* is a sequence of chained sowings which finishes either in the Ruma or in an empty hole.

The most elementary general result about winnability is the following:

Theorem 1. *For $n \geq 2$, the game cannot be won for*

$$k = \begin{cases} (n+1)^i, & i \geq 1; \\ n(n+1)^i, & i \geq 0. \end{cases}$$

Proof: In the first case, with $k = n+1$ (corresponding to $i = 1$), sowing from any hole goes once around the board and ends in the starting hole, for a loss.

For $k = (n+1)^i$ with $i > 1$, the first sowing from any hole goes around the board $(n+1)^{i-1}$ times and ends at that hole with $(n+1)^{i-1}$ seeds there, which the player must then sow again. By induction, the player eventually comes to sow from that hole with it containing $(n+1)$ seeds, for a loss.

In the second case, with $k = n$ (corresponding to $i = 0$), sowing from the leftmost hole ends in the Ruma and raises the number of seeds in each of the other holes to $(n+1)$. No matter which hole the player now chooses, the sowing goes around once and ends in that hole, for a loss. Sowing first from any hole but the leftmost results in the next sowing beginning from the previous hole, which now has $(n+1)$ seeds, and hence results in a loss. For $k = n(n+1)^i$ with $i \geq 1$, the first sowing goes around the board $n(n+1)^{i-1}$ times and ends at that hole with $n(n+1)^{i-1}$ seeds there, which the player must then sow again. By induction, the player eventually comes to sow from that hole with it containing n seeds. That sowing ends in the previous hole, which now contains the original $n(n+1)^i$ seeds plus others added with each sowing, for a total of

$$n(n+1)^i + n(n+1)^{i-1} + \cdots + n(n+1)^1 + n + 1 = (n+1)^{i+1}.$$

We are now in the situation of the first case, which leads to a loss. □

For convenience in further analysis, we introduce the following useful notations, derived from Backus-Naur form for describing the syntax of a language, for denoting the contents of holes. For X and Y sequences of digits representing the contents of consecutive holes,

- $\{X\}$ means 0 or more instances of X
- $\begin{pmatrix} X \\ \overline{Y} \end{pmatrix}$ means either X or Y

We put the Ruma at the right-hand end and separate it from the other holes by a bar. Thus, for example, the notation

$$2\{033\}033 \begin{pmatrix} 1 \\ \overline{2} \end{pmatrix} \mid 3$$

could denote any of the following game situations, with differing numbers of holes:

$$20331 \quad | \quad 3$$

$$20332 \quad | \quad 3$$

$$20330331 \quad | \quad 3$$

$$20330332 \quad | \quad 3$$

and so forth. We list the contents of the holes after each sowing and underline one position, which is either the hole that the last seed fell into or (if the last seed fell in the Ruma) the hole chosen to sow from on the next turn.

We exhibit in this notation two plays from the proof of **Theorem 1**. The play for $k = n + 1$ is represented by

$$\begin{array}{ccccc} \{n+1\} & \underline{n+1} & \{n+1\} & | & 0 \\ \{n+2\} & \underline{1} & \{n+2\} & | & 1. \end{array}$$

Similarly, the play for $k = n$ is given either by

$$\begin{array}{cccccc} \{n\} & n & \underline{n} & \{n\} & | & 0 \\ \{n+1\} & \underline{n+1} & 0 & \{n+1\} & | & 1 \\ \{n+2\} & \underline{1} & 1 & \{n+2\} & | & 2, \end{array}$$

which is a loss, or by

$$\begin{array}{cccc} \underline{n} & \{n\} & | & 0 \\ 0 & \{n+1\} & | & \underline{1}. \end{array}$$

For $n = 1$, we have won, but for $n \geq 2$, we must continue:

$$\begin{array}{cccccc} 0 & \{n+1\} & \underline{n+1} & \{n+1\} & | & 1 \\ 1 & \{n+2\} & \underline{1} & \{n+2\} & | & 2, \end{array}$$

which is a loss.

Values of k Close to n, or Larger than n

Although we do not expect to answer fully the general question of when this game is winnable, several interesting special cases invite mathematical analysis. One such case is the situation when k is close to n. For example, **Theorem 1** shows that the game is never winnable if $k = n$ or $k = n+1$ (for $n > 1$). Applying the notational analysis above to the case of $k = n + 1 - i$ gives the following theorem.

Theorem 2. *For $k = n+1-i$ with $i \geq 2$, the player's only chance is to start in one of the $i(i+1)/2$ holes farthest from the Ruma.*

Proof: When $i = 2$, we have $k = n - 1$. Remembering that there are n holes exclusive of the Ruma, the play goes as follows for a hole closer to the Ruma than the ones mentioned in the statement of the theorem:

$\{n-1\}$	$n-1$	$n-1$	$n-1$	$\underline{n-1}$	$\{n-1\}$	$\mid$	0
$\{n\}$	n	$\underline{n}$	$n-1$	0	$\{n\}$	$\mid$	1
$\{n+1\}$	$\underline{n+1}$	0	n	1	$\{n+1\}$	$\mid$	2
$\{n+2\}$	$\underline{1}$	1	$n+1$	2	$\{n+2\}$	$\mid$	3.

This sequence shows a forced loss if the player begins in any hole closer to the Ruma than the third hole from the left.

What happens for $i = 3$ and $k = n - 2$? Observe the play:

$\{n-2\}$	$n-2$	$n-2$	$n-2$	$n-2$	$n-2$	$n-2$	$\underline{n-2}$	$\{n-2\}$	$\mid$	0
$\{n-1\}$	$n-1$	$n-1$	$n-1$	$\underline{n-1}$	$n-2$	$n-2$	0	$\{n-1\}$	$\mid$	1
$\{n\}$	n	$\underline{n}$	$n-1$	0	$n-1$	$n-1$	1	$\{n\}$	$\mid$	2
$\{n+1\}$	$\underline{n+1}$	0	n	1	n	n	2	$\{n+1\}$	$\mid$	3
$\{n+2\}$	$\underline{1}$	1	$n+1$	2	$n+1$	$n+1$	3	$\{n+2\}$	$\mid$	4

This sequence shows a forced loss if the player begins in any hole closer to the Ruma than the sixth hole from the left.

This argument for $k = n-1$ and $k = n-2$ extends to prove the theorem. The sequence formed is 1, 3, 6, ..., the sequence of triangular numbers, the i^{th} triangular number being $i(i + 1)/2$. □

This theorem has content only if $n > i(i + 1)/2$, in other words, for n large compared with i, or put another way, n and k relatively close in size, e.g., approximately $k \geq n + 2 - \sqrt{2n}$.

The proof actually shows a little more than indicated by the theorem. If we do start in the $i(i+1)/2$ holes farthest from the Ruma, in many cases we still form the same sequence of holes with $n - i + 1$ seeds, then $n - i + 2$ seeds, and so on, until we have $n+1$ seeds in a hole, for a loss. This happens because the moves end successively rightward on the board, provided we miss ever landing in the Ruma. We hit the Ruma in this process only if we started from one of the holes i, $i + (i - 1)$, $i + (i - 1) + (i - 2)$, ..., $i(i + 1)/2$. In other words, there are only i holes to consider as the starting hole, not the $i(i + 1)/2$ holes that the theorem seems to suggest. Using this analysis, we can show the following:

Lemma 1. *For $k = n - 1$, the player can win only by starting from hole 3, followed by the second move again from hole 3.*

Although this result would seem to constrict the choices tremendously, the pattern of which games are winnable for $k = n-1$ is not clear. Computer analysis shows that the game cannot be won for n = 2–5, 7, 8, 11–12, and 22–23.

On the other hand, when $k = n - 2$, the situation appears clearer, although we cannot prove it. However, the following conjecture holds up to $n = 16$:

Conjecture 1. *For $n \geq 3$, the player can win the games with $k = n - 2$, except for the game with $n = 5$.*

In the other direction, for k slightly larger than n, we also know a little and have empirical evidence showing much more. We showed earlier that for $k = n+1$ or $k = n(n+1)$, the player cannot win. However, we conjecture:

Conjecture 2. *If $n + 2 \leq k < n(n + 1)$, then the player can win.*

This conjecture holds for $n \leq 12$. This is actually a weaker form of the following conjecture:

Conjecture 3. *If $k \geq n > 2$, and the game is not shown to be a forced loss by* **Theorem 1***, then the player can win the game except when $n = 3$ and $k = 6$.*

This conjecture is supported empirically by the values of n and k shown in **Table 1** later in the paper.

In general, it is fairly easy to win Tchuka Ruma when a relatively small number of seeds is involved. What seems to happen in most situations is that the game tree spreads out so quickly that within a few moves there are dozens (or hundreds) of possible plays of the game, at least one of which is winning. Thus, the only times that games seem to be non-winnable are when there are very tight constraints on the first few moves, producing a very narrow game tree and a fairly easy proof of nonexistence of a win. It may be that this is the only "reason" that these games are so often winnable, in which case it may not be possible to find constructive proofs of these wins. Probabilistic arguments might be easier to construct. It seems particularly telling that we have been able to find theorems that cover nearly all of the forced losses but as yet no theorems (apart from the case $n = 1$) to prove the existence of wins.

Small Values of k

Careful use of our notation and tedious consideration of cases, together with the recognition of patterns, lead to the following results. We give a proof of **Theorem 3** a little further on but omit the tedious proof of **Theorem 4**.

Theorem 3. *For $k = 1$, the player loses for all $n \neq 1, 3$.*

Theorem 4. *For $k = 2$, the player loses for all $n \neq 4, 7$.*

In addition, we make the following conjecture, based on extensive computer evidence:

Conjecture 4. *For any k, there is a value $K(k)$ for which the player loses for all $n \geq K(k)$. Furthermore, $K(k) \leq (k+1)^2$, i.e., if $n \geq (k+1)^2$, the player must lose.*

An alternative proof of **Theorem 3** is via the following lemma, which is interesting in its own right. For this lemma, we say that the *effective size* of a board is the number of holes (n) minus the number of leftmost holes that contain no seeds. Thus, the game with $n = 5$ and with holes (from farthest from the Ruma to nearest) containing 0, 0, 3, 1, and 0 seeds would have an effective size of three. This concept captures the fact that this game can be won using only the three holes closest to the Ruma, since no wrapping moves are required to win.

Lemma 2. *If a Tchuka Ruma configuration has either*

- *effective size $s \geq 6$ and fewer than s seeds outside the Ruma, or*
- *effective size $s \geq 11$ and at most s seeds outside the Ruma,*

then the game cannot be won.

Proof: In general, a game progresses in two stages. At first, some moves wrap around, but eventually no wrapping moves will be used again. During the first stage, the effective size of the board is not reduced by more than one (after the last wrapping move, it is possible for the formerly occupied hole farthest from the Ruma now to be empty, but not for both it and the next nearer hole to be empty). By induction, during the first stage the number of seeds is reduced by at least one (if this stage is nonempty) and the effective size is reduced by no more than one.

During the second stage, when no more wrapping moves occur, a single move can reduce the effective size of the board by two if the first two nonempty holes contain 1 and x seeds respectively, with $x \neq 0$. Consider the hole adjacent to the Ruma. If it has more than one seed in it, any play from it must wrap. Every move that does not start from that hole puts one seed there, hence the next move must remove that seed by playing from that hole. Consequently, every second move must be a play from the hole adjacent to the Ruma. Thus, in every two moves the effective size of the board can be reduced by at most two, and induction again applies.

To finish the proof the lemma, the major difficulty is in establishing the base case. Since the induction in the second stage proceeds by reducing the effective size by two, we must establish a base case for two successive values of n. For the first case of the lemma, an easy analysis shows that the statement holds when $n = 6$ and $n = 7$. A computer analysis shows that the second case of the lemma holds when $n = 11$ and $n = 12$. □

Since it is easy to verify **Theorem 3** for $n < 11$, this lemma establishes the theorem for all n. We suspect that the true lower bounds on the number of seeds in a winnable configuration grows like $nf(n)$, where $f(n) \to \infty$. If this is true, it would establish the first part of **Conjecture 4**, but we do not have enough evidence for this suspicion to justify setting out a conjecture.

Small Values of n

The Pattern for $n = 1$

For $n = 1$—just one hole besides the Ruma—there is just one hole to play from, and play continues until all seeds are in the Ruma or else a turn ends with a single seed falling in the single non-Ruma hole. The only way that the latter can happen is for the turn to have begun with two seeds in the non-Ruma hole and all the rest in the Ruma. A game must come down to either

$$\begin{array}{ccc} \underline{2} & | & k-2 \\[1em] 1 & | & k-1, \end{array}$$

which is a loss, or else to

$$\begin{array}{ccc} 1 & | & \underline{k-1} \\[1em] 0 & | & k. \end{array}$$

Starting from the initial position, we have one of two results. If k is even, we get

$$\underline{k/2} \quad | \quad k/2;$$

but if k is odd, we get

$$(k-1)/2 \quad | \quad \underline{(k+1)/2}$$

The easiest way to see what must happen is to express k in binary notation. Then the first kind of result corresponds to stripping off a trailing

0; the second corresponds to stripping off a trailing 1. For example, with $k = 11$, writing the contents of the first hole in binary but the contents of the Ruma in decimal, we have:

$$\begin{array}{r|r} 1011_2 & 0 \\ 101_2 & 6 \\ 10_2 & 9 \\ 1_2 & 10 \end{array}$$

Reducing k in this fashion eventually produces either 10_2, corresponding to 2 seeds in the non-Ruma hole and hence a loss, or else 11_2, which reduces to 1 and corresponds to 1 seed in the non-Ruma hole and hence a win.

The player wins for $k = 1$, 3, 6–7, 12–15, 24–31, 48–63, 96–127, ..., and loses for $k = 2$, 4–5, 8–11, 16–23, 32–47, 64–95, ..., which we put more formally as follows:

Theorem 5. *For $n = 1$: Let $i = \lfloor \log_2 k \rfloor$. The player wins if and only if*

$$k \geq 2^i + 2^{i-1} = 3 \cdot 2^{i-1}$$

(equivalently, if and only if $4 \cdot 2^{i-1} > k \geq 3 \cdot 2^{i-1}$).

The "Sporadics"

We give in **Table 1** the values of n and k for which the player must lose.

The underlined values of k in the table are not explained by the theorems that we have developed so far. We refer to such values as *sporadics*.

The Case $n = 2$

The situation for $n = 2$ is surprisingly complicated. No simple pattern of when the game can be won is apparent. In fact, as indicated by **Conjecture 3**, this is the only situation (save $n = 3$, $k = 6$) where there appear to be losing games with $k > n + 1$ that are not required by **Theorem 1**, and this is indicated in the table of sporadics.

Open Problems

Table 1 leads to several obvious questions, including those addressed by our earlier conjectures. In particular:

Table 1.
Values of n and k for which the player must lose, as determined by computer analysis. Underlined values are not explained by the theorems.

Holes n	Seeds k	Largest k tested
1	2, 4–5, 8–11, 16–23, 32–47, 64–95, ...	all
2	1–3, 6, 9, <u>11</u>, 18, <u>20</u>, 27, <u>30</u>, 54, 81, 162, <u>168</u>, 243, 486, 729, 1458, 2187, 4374, 6561, 13122, 19683, 39366, 59049	100,000
3	2–4, <u>6</u>, 12, 16, 48, 64, 192, 256, 768, 1024, 3072, 4096, 12288, 16384, 49152, 65536	100,000
4	1, <u>3</u>–5, 20, 25, 100, 125, 500, 625, 2500, 3125, 12500, 15625, 62500, 78125	100,000
5	1–<u>3</u>, <u>4</u>, 5–6, 30, 36, 180, 216, 1080, 1296, 6480, 7776, 38880, 46656	100,000
6	1–<u>3</u>, 6–7, 42, 49, 294, 343, 2058, 2401, 14406, 16807	100,000
7	1, <u>3</u>, <u>4</u>, <u>6</u>, 7–8, 56, 64, 448, 512, 3584, 4096, 28672, 32768	100,000
8	1–2, <u>4</u>, <u>7</u>, 8–9, 72, 81, 648, 729, 5832, 6561	10,000
9	1–2, 9–10, 90, 100, 900, 1000	1,000
10	1–2, 10–11, 110, 121	1,000

- For $k < n$, can we detect any pattern to the lost games? **Conjectures 1** and **4** partly address this problem.
- For $n = 2$, are there any sporadics other than the ones in this table?
- For $k > n > 2$, can we establish **Conjecture 3**? Or even the weaker **Conjecture 2**?

How Many Positions Are There?

An argument using geometric series shows that a hole can never contain more than $k + \frac{1}{2}k(n-1)$ seeds. Using this bound, and applying occupancy theory, there can be no more than

$$\binom{n(k+1)}{n} - (n-1)\binom{\frac{nk}{2}+n-\frac{k}{2}-1}{n}$$

positions reached in the game. Neither bound is very tight for moderate n and k. For example, for $n = 3$ and $k = 10$, our tree program examines 824 positions, while our bound is $5456 - 2 \times 220 = 5016$.

Variations

Here we describe and discuss several variations on the basic game of Tchuka Ruma, which come purely from our own imaginations.

Tchuka Ruma without Wrapping

In basic Tchuka Ruma, play *wraps* around the Ruma. An interesting variation is to prohibit wrapping and sowings that would involve it. Play ends when the player lands in an empty hole or can no longer make a permitted sowing, and the object of the game is to get *as many* seeds into the Ruma as possible. It is impossible to get seeds into the Ruma from the $(k-1)$ holes closest to the Ruma, so a sowing that ends in one of the closest k holes ends the game. Investigating this variation of Tchuka Ruma seems valuable because of its connection with the proof of **Lemma 2**, although in that proof it is necessary to allow the number of seeds in each hole to vary. Broline and Loeb [1995] have been very successful in analyzing this form of the game with the additional limitation of no chaining of the sowings, i.e., a move consists of a single sowing, even if the sowing ends in a nonempty hole.

It would seem reasonable that the more holes, the more seeds the player can eventually play into the Ruma. Our evidence suggests otherwise, however, and we have the following very surprising conjecture:

Conjecture 5. *Given k, there is an upper bound $U(k)$, such that no matter how large n is, the player can never get more than $U(k)$ seeds into the Ruma.*

For small values of k, we have tested this conjecture up to substantial values of n. Let $\max(k)$ be the largest number of seeds that can be placed in the Ruma. For $k = 1, \ldots, 4$, **Table 2** gives the apparent value of $\max(k)$, the smallest value of n for which this value is first achieved, and the largest value of n through which we have tested.

Table 2.
Apparent values for $\max(k)$, the largest number of seeds that can be placed in the Ruma when starting with k seeds in each hole.

k	$\max(k)$	reached for $n =$	tested up to $n =$
1	7	11	75
2	15	20	75
3	13	24	75
4	26	35	50

Two-Person Tchuka Ruma with One Hole

We do *not* refer here to any of the hundreds of variants on the usual two-person mancala game, in which each player has a Ruma and the object is to put more seeds into one's own Ruma than the opponent can put into the opponent's Ruma.

Instead, we try to preserve essential features of Tchuka Ruma. As in Tchuka Ruma, a player's turn continues, chaining from one sowing to another, except that the turn ends when a sowing lands in the Ruma. The other player may then play from any hole (except the Ruma). A sowing that ends in an empty hole (except the Ruma) is prohibited; in actual play, such a move loses the game, since a player ought to anticipate where a turn will end (and usually board games do not allow a move to be taken back). A player whose turn it is, but who cannot make a legal play, loses. The game may be played with or without wrapping permitted. One may also consider the *misère* version, in which the first player unable to make a legal move wins.

This game fits the framework of the two-person impartial games discussed in Berlekamp et al. [1982] and hence can be analyzed by the techniques exposited there.

Programming Considerations

We have found that writing a program to search for wins in this game is an excellent example of recursion for introductory computer science courses. To decide if a given position can be won, successively sow from each hole and recursively see if the resulting game can be won. To sow, sow the seeds from one hole, and if the sowing ends in a nonempty hole other than the Ruma, recursively sow again.

The sowing recursion is tail-end recursion, which can easily be converted to a loop (doing so speeds up our program by up to 21%). The use of recursion to decide winnability, however, cannot easily be removed; so the game gives an example of "real" recursion that does not use sophisticated data structures (to complement the traditional Towers of Hanoi example.)

In a more advanced computer science course, the game is an excellent example to demonstrate dynamic programming. In analyzing all games for a fixed n, the same endgame (with only a few seeds remaining) can arise a great many times (and even more as different values of k are investigated). In dynamic programming, we keep track of the endgames already seen and whether they are wins or losses.

With small values of n, say $n = 2$ or 3, we can keep the "known" endgame configurations in an n-dimensional array. Doing so is equivalent to the simpler forms of dynamic programming that are often taught in computer science courses (e.g, calculating Fibonacci numbers). For larger n, we instead encode endgames as strings and store them in a binary tree. With either implementation of dynamic programming, we get speedups of 30–60%. The increased speed allows us to investigate larger values of n and k than we could without dynamic programming, although we still quickly run into the limits of computational ability.

Appendix: Anthropological Details

Where Did It Come From?

Solitaire Tchuka Ruma also involves another mystery: Where did it come from? We quoted in full the problem and solution given by Degrazia [1948] because it is the sole reference cited by both Averbach and Chein [1980, 312, 395] and Bell and Cornelius [1988, 33, 37, 110] in their presentations of the game, and those authors have written us that they do not know anything further about the game's origin [Bell 1992; Cornelius 1992]. Degrazia does not cite any specific source. Averbach and Chein refer to the "modern game" of "Ruma," perhaps a trademarked commercial game, but we have been unable to trace one.

We note below possible origins of the solitaire game. It is possible that it was invented independently in several locations. We think, however, that there may be a single recent common source for the game with the specific rules that we have quoted.

Maybe from Egypt, Long Ago?

Two-player games of mancala type, on a board with two, three, or four rows, are played in most of the world:

> Boards of the two row type are known from Egypt in the period from c. 1580–1150 B.C. It next appears in Ceylon in the first centuries A.D.

> and in Arabia before Muhammad (Murray, p. 159). It is currently played in the Philippines, Indonesia, Malaysia, southern China and mainland South-East Asia, south India and Ceylon, the Maldive Islands, Madagascar, the west coast of Arabia, most of Africa, and was introduced by the slave-trade into the Caribbean, regions on the east coast of South America and in the southern part of the United States (Béart, p. 480; Murray, p. 158).
>
> ... There is a consensus ... that the game has spread from west to east in Asia and from the northeast to the west and south in Africa. [Barnes 1975, 71]

Deledicq and Popova give a map of the diffusion of mancala games [1977, 20]. They consider possible origins of the games, dismissing Egypt and Sumeria, and conclude with the belief that the games originated in Africa, with the Arabs playing a role in the diffusion [1977, 30].

Maybe from Southeast Asia?

Degrazia cites an "East Indian" origin for Tchuka Ruma.

Games played by the Achèhnese of northern Sumatra

Kaudern [1929, 316], quoting from the *Enc. v. Ned. Indië*, gives *tjoeka* and *djoenka* as names for a game among the Achèhnese of northern Sumatra. The game, "much played by women and children," is known in different places in Achèh as *chatō* ("also the name for the ordinary game of chess"), *chuka'*, and *jungka'* [Snouck Hurgronje 1906, 200–201]. Snouck Hurgronje names four different ways of playing the game in Achèh but describes only one, *meusuëb*. This version, played with two rows of six holes, has virtually the same rules as *motiq*, which is played with two rows of seven holes by the Kédang people on the island of Lembata (Lomblen) in the Solor Archipelago, near Timor [Barnes 1975, 75–78]; Russ [1984, 65–66] describes both versions. The names *sungka, sunca, chonka,* and *chongkak* are used for variations played in other parts of Indonesia and the Philippines [Russ 1984, 64].

The four variations named by Snouck Hurgronje are described in detail in the entry for "tjatō" in a dictionary of the Achèh language [Djajadiningrat 1934].

- *meusoeëb*: If one's last seed lands in an empty hole, one's turn ends. One captures by having one's last seed make a total of four in a hole on one's own or on the opponent's side of the board and removing the four. (Djajadiningrat notes that *tjatō* comes from the Sanskrit word *catur* for "four.") One's turn then ends if the following hole is empty; otherwise, one takes the seeds from that following hole and continues with them.

- *meuta'*: One captures by having one's last seed land in an empty hole on one's own side, removing the contents from the hole opposite. One also "captures" that opposite hole; both players should skip that hole on subsequent sowings, and any seeds put into it by mistake are captured.

- *meutjōh*: (This two-person game is the closest of the four to the rules of Tchuka Ruma.) If the last seed lands in one's own *tjōh* (what Degrazia calls the Ruma), then one may play again from any of the holes on one's side of the board. If one's last seed lands in an empty hole on one's own side of the board, one captures the contents of the opponent's hole opposite plus the last seed itself and one's turn ends.

- *meuliëh*: If one's last seed lands in a nonempty hole, one leaves it there and continues sowing with the contents of the following hole. If one's last seed lands in an empty hole that is followed by two empty holes, then one's turn ends. If one's last seed lands in an empty hole that is followed by just one empty hole, one continues sowing with the contents of the hole following the empty hole. If one's last seed lands in an empty hole that is followed by a nonempty hole, one captures the contents of the nonempty hole and continues sowing with the contents of the hole after that.

The wide variety of these rules is remarkable.

The origin of the words "Tchuka" and "Ruma"

It is challenging to try to determine the etymology of the name "Tchuka Ruma," which could give some clue to the game's origin.

"Rumah" means "house" in Malay and in Indonesian. The holes are called *rumòh* / *roemòh* by the Achénese, according to Snouck Hurgronje (writing in English) and Djajadiningrat (writing in Dutch). The word *rumah* is applied in some cultures to refer only to the two holes at each end of the board where the players accumulate seeds that they have won, consistent with Degrazia.

The definitive etymology of *chuka* in Malay (*cuka* in Indonesian) is from the Sanskrit *chukra* (fruit vinegar made from the tamarind) [Behrend 1995]. In Malay and in Indonesian, the word means "vinegar" and likely has nothing to do with mancala games (unless tamarind seeds were used as the playing seeds, as they commonly are in games in India).

The name in Malay for two-person mancala games is *chongkak*, which may come from Malay words

- *jong*, from the Chinese junk-like shape of the board;

- *chongkak*, the name for the cowrie shells with which the game is played (this is also the name of the game on the island of Ceylon); or

- *cekak*, which means “to grasp.”

For example, in Hellier [1907] we find:

> Haji Othman the Visiting Teacher of Province Wellesley ... describes Jongkak as a women’s game originally played by the ladies at the courts of the Malay Rajas. The playing board is shaped like a junk or boat, and, according to Haji Othma, the name of the game is derived from ”jong” a junk. ...
>
> The game is for one or two people. ... [A seed falling into the large hole at the end] is said to have ”entered the house” (naik rumah). ...

Contemporary inquiries

Our students from Indonesia and the Philippines all are familiar with a two-row two-person game with six or seven holes per row. One student remembers his grandfather from China playing a solitaire game with one row in Sumatra [Suroto 1993]. A student from Mali remembers his aunt N’Paï Diarra playing a solitaire form of mancala shortly before her death in 1981 at age 92 [Diarra 1995].

Another Beloit student from Indonesia [Sutjipta 1995] reports an interesting two-person variation of Tchuka Ruma called “Dampu.” In this game, a two-row mancala-style board is used, with a varying number of holes in the rows. Each player begins from any hole on that player’s own side of the board and follows the rules for Tchuka Ruma, playing seeds into the player’s own Ruma when appropriate and passing in circular fashion around the entire board, but not playing seeds into the opponent’s Ruma. The two players play simultaneously; but the player who finishes a move first must wait for the opponent to finish. Both players then simultaneously make another move, continuing in this fashion until no more seeds are available. At that point, the player whose Ruma contains more seeds wins. Our student reports that when she cannot find an opponent, she will sometimes play the game as a solitaire, using just one side of the board. Doing so turns the game into Tchuka Ruma solitaire, with the object of getting as many seeds as possible into the Ruma in as few moves as necessary.

Inquiries posted to various `soc.culture` Internet newsgroups (the ones for `indonesia`, `filipino`, and `malaysia`) have produced little further information. There was one encouraging response from a Filipino woman:

> I do remember my late grandmother playing by herself, the sungka board on her lap, with just one of the big holes nearly overflowing with shells. She probably was using only half the board because the way it was tipped over on her lap, shells would have spilled out if the other half was being used as well. [Blanco 1994]

Association of the game with wakes

In some cultural groups, the game of mancala type is played only at wakes [Barnes 1975], a fact that was confirmed by one of our correspondents:

> I just bought the board when I went home for Christmas. 7 holes, 7 stones per hole. The rules [for the two-person game] are similar to how you describe it. . . . I am not an expert on sungka but I have seen players during wakes playing seemingly forever. By the way, sungka is generally played during wakes and it is a superstition (at least in my home town) that it is bad to play it outside of the wake situation. . . . I'm from Malinao, Aklan in Panay Island, Visayas, Philippines. When I bought my board, I also bought one for my nephews . . . and proceeded to teach the kids. Their 'yaya' (nursemaid) forbade them to play because it was bad luck until she realized of course that we just buried my mother, so it was OK to play. [Ilio 1994]

Spiritual significance of mancala games

We quote from Lokman [1995]:

> The game has acquired a spiritual significance among the people of the region. It has helped to foster the belief that ultimately, the fate of an individual may not be decided by that person. There are other forces in the universe which manipulate and set patterns for what has transpired, what is happening now and what will occur in the future. If you look at other Malayo / Indonesian art forms such as the Wayang (Shadow Puppetry), you can see the same motifs.
>
> As with all other aspects of Malay culture, the chongkak has acquired its own supernatural aspects. I would like to remind you at this point that the ancient Malays practiced a very complex form of shamanism. Through the subsequent centuries, the shaman / animistic religion was combined with Hinduism, Buddhism, Confucianism and finally Islam. The significance of the chongkak may be explained by these traditions. There are still mystical circles today that believe the chongkak to be a sort of ouija board. When one is playing the game, other netherworldly forces are concentrated on or around the board. The forces of good and evil in some way affect the movements on the board. For this reason games are usually prohibited on Fridays (the Muslim Sabbath). I rather suspect that some of these prohibitions were engineered by older folk to control their younger kin. After all, too much of anything may be fatal.

Or Maybe from Russia???

The game of "Tchouka" and its "Rouma" is first mentioned in mathematical literature by H. Delannoy [1895], who gives no provenience or explanation of the terminology. He does give specific rules for the game as a solitaire (only), which are exactly the same as those of Degrazia, and notes that his contribution is from his correspondence with Édouard Lucas, the famous mathematician of recreations.

Later, Sainte-Laguë says that it is a Russian game "à peu prés complètement inconnu chez nous" ("virtually unknown among us") [1929, 42; 1937, 99]. He repeats the rules of Delannoy.

Deledicq and Popova mention the two-person game as played among the Kazaks in northwest Mongolia and among the Kirghiz under the respective names *Toguz korgol* and *Eson xorgol* ("game of nine dung balls") [1977, 40, 84–87]. They also note the game's presence in Uzbekistan, Tajikistan, Turkmenistan, and Afghanistan [1977, 40].

In Deledicq and Popova we also find the game "La Tchouka," described as a game that several people can play but in principle a solitaire game [1977, 99–100]. They

- give the same rules as in Delannoy;
- show diagrams of linear and circular boards;
- note that the game was formerly spread through Russia but its name is not Slavic, and "suppose" that the name was borrowed from paleosiberian and eskimo populations, who play games of skill and chance under the same name (*chuki*); and
- use the term *rumba* for the last hole.

Deledicq and Popova include an essay by Véronique Gautheron in which she offers some imagined variations on the solitaire game of Tchouka and does some mathematical analysis of them [1977, 180–187]. She cites Sainte-Laguë [1937], and she uses the term "roumba." She distinguishes variations as follows:

Le Tchoukaillon: A solitaire game, played like Tchuka Ruma, except that

- the player decides how many seeds to put in which holes,
- a player may play only from a hole for which the move lands exactly in the Ruma, so there is
  - no "chaining" of sowings: the last seed of a sowing always lands in the Ruma, never in any other hole, whether empty or full; and
  - no wrapping: sowing is not allowed to circle through the Ruma and back to the other side of the board.

Gautheron outlines a proof that for each number of seeds, there is a unique way to distribute them so as to win the game; the proof proceeds by uniquely reversing the moves of the game. She notes that the strategy for winning is, from among playable holes, to choose the one closest to the Ruma. She asks for an easy way to find winning positions and for the number of holes used by a winning position with n seeds. This is the game analyzed by Broline and Loeb [1995], and they answer those questions.

Le Monokalah. Like Tchoukaillon, with wrapping but no chaining. She observes for the game with four holes that the player cannot win with 10 or 12 seeds, and that there is never more than one way to distribute the seeds so as to be able to win. She asks what positions are winnable and with what strategy.

La Multitchouka. Like Tchoukaillon but with both wrapping and chaining, so the rules of play are exactly the same as Tchuka Ruma. The difference is that in Multitchouka, the player decides on the initial distribution of the seeds among a fixed number of holes. Gautheron gives a game tree of winnable positions for three holes and ten seeds, noting that successive positions in a game form a partially ordered set.

Our Conjecture

The combined evidence leads us to suspect that the solitaire game originated in the Malaysia / Indonesia area and that the game and its name may have traveled to Russia (though we have not yet been able to confirm the presence in Russia). This opinion is not shared by all; we quote from Lokman [1995]:

> I rather suspect that it was the other way around. The peoples of Malaya and Indonesia possess very little in their culture which may be considered native. Apart from the old shamanism (which I suspect may have originated from the beliefs of the proto-Mongolian ancestors of the Malays), everything else has been borrowed. The Malay language is itself a combination of Sanskrit, early Chinese dialects, Arabic, Urdu, and more recently English. In addition to this, there are hints in the grammar and vocabulary of Portuguese, Dutch, Persian, Turkish, and Japanese influences.
>
> It is conceivable that this game was brought by traders from the Altaic / Central Asian Region more than half a millenium ago. Although Malaya and Indonesia were never part of the Mongol Empire, there are records that show tribute being sent to the Great Khans and show Mongol envoys and merchants finding their way to Malacca and

other ports. It is certain that contacts were maintained with the Imperial Yuan Dynasty in China and the Ilkhanate in Persia throughout their lifetimes. Contacts with Russia would have come about through various dealings with the Khans of the Golden Horde. After all, the Malayan-Indonesian region was the major source for spices in the world. Up until two centuries ago, it was still the major commercial area in the world.

Nevertheless, the preponderance of historical and linguistic data, along with the continued existence of several games closely related to Tchuka Ruma, still suggests to us a Malaysian / Indonesian origin of this game.

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