PERIODIC TILINGS AND TILINGS BY REGULAR POLYGONS

by

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PERIODIC TILINGS AND TILINGS BY REGULAR POLYGONS

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Under the supervision of Professor Donald W. Crowe

Abstract: We assume a tiling has, under its symmetry group, \( v \) orbits of vertices; \( e \) orbits of edges; and \( t \) orbits of tiles. Inequalities are established relating these parameters, both for arbitrary tilings and for tilings by regular polygons, and we show that some of these inequalities are sharp. In the case of tilings by regular polygons, we classify those tilings with \( v \leq 3 \), \( e \leq 3 \), or \( t \leq 2 \), and show that the number of tilings with some fixed number of orbits of vertices [or edges; or tiles] is finite. The edge figures which can occur in a tiling by regular polygons are classified, as are tilings which contain at most three different types of these edge figures.

Progress is made towards classifying those tilings by regular polygons which contain at most two different types of vertex figures.

With respect to tilings by regular polygons which contain only two types of tiles (two congruence classes of polygons), the number of possible orbits of each polygon is determined. Tilings by regular polygons in which any two congruent tiles are equivalent under the symmetries of the tiling are classified, as are tilings which satisfy a similar condition on the edges.
"We're all in it - we're all tilled, here."

   Olga,

"He's got 'em on the list - he's got 'em on the list;
And they'll none of them be missed - they'll none of them
be missed."

   Chorus of Men,
   The Mikado, by Gilbert and Sullivan.

Dedicated to the two women I love
   Peggy and Eunice Chavey.
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Now that it's almost over, it seems amazing to me that my friends and my thesis committee (which are not exclusive) have managed to put up with me for the last month or so. They are among the many people I wish to thank for helping to make this thesis possible.

None of this work would have been possible without the excellent survey of the subject by Grünbaum and Shephard, and I wish to thank them for making their advance copy available to us. Professors Donald Crowe and Michael Bleicher deserve thanks for their efforts in creating and sustaining a seminar covering this work, and it was from this seminar that most of these results developed.

Much of the work in this thesis owes a great deal, in ways that are difficult to pin down, to conversations with Don Crowe and Mike Bleicher; but some of the work can be more directly attributed to my colleagues. Mary Leland discovered one class of tilings used in the proof of theorem 2.3 (as mentioned there), and this class helps to extend the known range of realizable parameters in tilings. The nice proof of fact 1 in section 1.3 is a drastic improvement of my original, and this proof was pointed out by John Rosenberg. Elsa Gunter volunteered to draw most of the tilings in figures 5.2 - 5.5 on a Carnegie-Mellon laser printer, and these figures (one of prettier aspects of the thesis) would have been impossible without her help.
The remaining tilings in these figures were constructed by hand, mostly through the aid of Peggy Chavey, and I am indebted to her for this (among other things). Figures 3.1, 4.1, and 4.2 are all taken from Grünbaum and Shephard's works.

From a more personal standpoint, I would like to thank those people who have been instrumental in my mathematical development. First and foremost among these is my mother — who encouraged me from a young age, and first introduced me to such topics as projective geometry and number theory. If my memory serves me correctly, she also is the one who introduced me to H.S.M. Coxeter, who is now my academic grandfather. I must also mention Leonard Forbush, a high school instructor and a strong influence on me both mathematically and personally; and Prof. William V. Caldwell (Univ. of Mich.-Flint), who once told me "You can't be a promising young mathematician all your life." Memories of his admonishments to me have helped a great deal in preventing my extra-curricular activities from overwhelming my mathematics.

More recently, I wish to thank the friends who have helped me maintain sanity and willpower during this recent struggle to finish. Peggy, my wife, who along with her much needed personal aid and support, also drew figures, proofread the thesis, corrected unintelligible passages, and generally gave up her summer to help me. Mary and Will Leland have been great friends in a bind, especially Mary,
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Finally, I cannot thank Don Crowe, my advisor, quite enough for all the help he has been. At times, it seems he was spending as much time on my work as I was; and he contributed a great deal to helping my writing and improving both my mathematics and my mathematical exposition. Throughout my graduate career, he has introduced me to many fascinating areas of mathematics and has always been willing to help or just to listen. I am very pleased to have been able to finish my degree under him.
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